

$Sp(4, \mathbf{R})/GL(2, \mathbf{R})$ Matrix Structure of Geodesic Solutions for Einstein–Maxwell–Dilaton–Axion Theory

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Abstract

The constructed $Sp(4, R)/GL(2, R)$ matrix operator defines the family of isotropic geodesic containing vacuum point lines in the target space of the stationary D=4 Einstein–Maxwell–dilaton–axion theory. This operator is used to derive a class of solutions which describes a point center system with non-trivial values of mass, parameter NUT, as well as electric, magnetic, dilaton and axion charges. It is shown that this class contains both particular solutions Majumdar–Papapetrou–like black holes and massless asymptotically nonflat naked singularities.

I. INTRODUCTION

In recent years much attention has been given to the study of gravitational models appearing in superstring theory low energy limit [1]– [5]. Einstein–Maxwell theory with dilaton and axion fields (EMDA) is one of such models. It appears in the frames of heterotic string theory as a result of omission of a part of vector and scalar fields arising during extra dimensions compactification. As it has been established earlier, the theory under consideration

leads to three-dimensional σ -model with symmetric target space which possesses an isometry group locally isomorphic to $Sp(4, R)$ [6], [7], and the model admits a null-curvature $Sp(4, R)/U(2)$ coset representation [8]– [10]. The brief description of matrix formalism is given in the following section.

Subsequently the class of exact solutions to the motion equations written in matrix form is constructed. Using the Kramer–Neugebauer approach [11], we consider coset space matrix M dependance on one space coordinate function $\lambda(x^i)$. The found solutions are corresponding to isotropic geodesic lines family in the target space and to the set of point centers in the coordinate three-dimensional space. In case of magnetic, axion and NUT charges absense the represented class transforms into the earlier obtained by Gibbons [12]. The connection with other known special solutions [14], [15] is established during the study of the case of null-curvature matrix M linear dependance on the function λ . Then Majumdar–Papapetrou-like black hole solutions family and massless naked singularities can be obtained from the general one. A complete list of these particular solutions is given in the last section of the article.

II. MATRIX REPRESENTATION OF THE STATIONARY STRING GRAVITY EQUATIONS

Let us discuss low energy effective four-dimensional action, which describes the bosonic sector of the heterotic string, taking into account the contribution of gravitational, Abelian vector, dilaton and axion fields:

$$S = \int d^4x |g|^{1/2} (-R + 2\partial\phi^2 + \frac{1}{2}e^{4\phi}\partial\kappa^2 - e^{-2\phi}F^2 - \kappa F\tilde{F}), \quad (1)$$

where $R = R^{\mu\nu}_{\dots\mu\nu}$ is the Ricci scalar ($R^{\mu}_{\dots\nu\lambda\sigma} = \partial_\lambda\Gamma^{\mu}_{\dots\nu\sigma}\dots$) of the 4-metric $g_{\mu\nu}$, signature $+- --$, $\mu = 0, \dots, 3$ and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}E^{\mu\nu\lambda\sigma}F_{\lambda\sigma}. \quad (2)$$

In doing so we consider that the scalar field ϕ is the dilaton one, and the axion is written in the form of pseudoscalar field κ .

As it has been done [11], [16], the four-dimensional line element can be parametrized according to

$$ds^2 = f(dt - \omega_i dx^i)^2 - f^{-1}h_{ij}dx^i dx^j, \quad (3)$$

where $i = 1, 2, 3$. Below we will study the stationary case when both the metric and the matter fields are time independent. It has been shown before [6] that in this case part of the Euler-Lagrange equations can be used for the transition from both spatial components of the vector potential A_i and entered in (3) functions ω_i to the magnetic u and rotation χ potentials respectively. The new and old variables are connected by differential relations:

$$\nabla u = f e^{-2\phi}(\sqrt{2}\nabla \times \vec{A} + \nabla v \times \vec{\omega}) + \kappa \nabla v, \quad (4)$$

$$\nabla \chi = u \nabla v - v \nabla u - f^2 \nabla \times \vec{\omega}. \quad (5)$$

The new notation $v = \sqrt{2}A_0$ is entered and the three-dimensional operator ∇ is corresponded to the three-dimensional metric h_{ij} . Also it has been found that expressed in terms of $f, \chi, u, v, \phi, \kappa$ variational equations for the action (1) are at the same time Euler-Lagrange equations for the three dimensional action

$${}^3S = \int d^3x h^{\frac{1}{2}}(-{}^3R + {}^3L), \quad (6)$$

where 3R is the curvature scalar constructed according to 3-metric h_{ij} and

$$\begin{aligned} {}^3L = & \frac{1}{2}f^{-2}[(\nabla f)^2 + (\nabla \chi + v \nabla u - u \nabla v)^2] \\ & - f^{-1}[e^{2\phi}(\nabla u - \kappa \nabla v)^2 + e^{-2\phi}(\nabla v)^2] \\ & + 2(\nabla \phi)^2 + \frac{1}{2}e^{4\phi}(\nabla \kappa)^2 \end{aligned} \quad (7)$$

Thus in the stationary case the string gravity appears to be the nonlinear σ -model. As it was shown [6]– [9], the three-dimensional Lagrangian 3L is invariant under the ten-parametric continuous transformation group isomorphic to $Sp(4, R)$. Then it was established that 3L can be rewritten with the aid of the four-dimensional matrix M in the form

$${}^3L = \frac{1}{4}Trj^2, \quad j = \nabla M M^{-1}, \quad (8)$$

and M , being the matrix of the coset $Sp(4, R)/U(2)$, satisfies the symplectic and symmetric properties,

$$M^T J M = J, \quad M^T = M, \quad (9)$$

where

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \quad (10)$$

It's easy to see that any symplectic matrix G defines automorphism $M \rightarrow G^T M G$ for the coset under consideration. The relations (9) allow to parametrize the matrix by six independent functions which can be chosen as potentials $f, \chi, u, v, \phi, \kappa$. Here it is convenient to use the two-dimensional matrices P and Q which define the Gauss decomposition

$$M = \begin{pmatrix} P^{-1} & P^{-1}Q \\ QP^{-1} & P + QP^{-1}Q \end{pmatrix}. \quad (11)$$

Their evident form is [10]:

$$P = \begin{pmatrix} f - v^2 e^{-2\phi} & -v e^{-2\phi} \\ -v e^{-2\phi} & -e^{-2\phi} \end{pmatrix}, \quad (12)$$

$$Q = \begin{pmatrix} -\chi + vw & w \\ w & -\kappa \end{pmatrix}, \quad (13)$$

where $w = u - \kappa v$.

III. THE GENERAL GEODESIC ISOTROPIC SOLUTION

The appropriate to the three-dimensional action motion equations have the standart form

$$\nabla j = 0, \quad (14)$$

$${}^3R_{ik} = \frac{1}{4}Tr(j_i j_k), \quad (15)$$

and admit the procedure of exact solution construction stated before by Kramer and Neugebauer for arbitrary σ -models [11] and developed later by Clement for the case of $SL(3, R)/SO(2, 1)$ matrix representation of Kaluza–Klein five-dimensional theory [17]–[19]. We consider the ansatz for which the matrix M is determined by the aid of one space coordinate function λ

$$M = M(\lambda), \quad \lambda = \lambda(x^i), \quad (16)$$

when $\lambda(x^i)$ is supposed to satisfy the Laplace equation:

$$\nabla^2 \lambda = 0. \quad (17)$$

It is not difficult to prove that the ‘material’ equation (14) turns into a relation, determining the dependance of M on λ :

$$\frac{d}{d\lambda} \left(\frac{dM}{d\lambda} M^{-1} \right) = 0. \quad (18)$$

The sense of (18) becomes clear after the introduction of the so called target space, i.e., of the metric space with the coordinates $f, \chi, u, v, \phi, \kappa$ and the linear element uniquely connected with the Lagrangian 3L :

$$dS^2 = \frac{1}{4}Tr(dMM^{-1}dMM^{-1}). \quad (19)$$

Then the formula $M = M(\lambda)$ determines a line in the target space which according to (18) is a geodesic one [11].

The solution of the equation (18) is

$$M = SM_0, \quad (20)$$

where

$$S = e^{\lambda A} = \sum_0^{\infty} \frac{(\lambda A)^n}{n!}, \quad (21)$$

$A = \text{const}$ and $M_0 = M|_{\lambda=0}$. The matrix S , so far is only formally determined, later on by natural causes it will be called the evolutionary operator.

The three-dimensional Einstein equations (15) in view of (20) and (21) can be rewritten

$${}^3R_{ik} = \frac{1}{4} \lambda_{,i} \lambda_{,k} \text{Tr} A^2 \quad (22)$$

and form together with (17) the complete system of equations which determines the three-dimensional metric h_{ik} and the scalar function λ .

Let us now establish the conditions for the evolutionary operator S and the matrix A determined by it. Their fulfilment ensures that M belongs to the coset space $Sp(4, R)/U(2)$ along the whole geodesic line, only if it is true for $\lambda = 0$, i.e., for the matrix M_0 . It is evident that if the matrices M_0 and M are symplectic ones, the operator S should possess the same feature, and it is convenient to rewrite the first of the relations (9) for it in the form of

$$S^T = -JS^{-1}J, \quad (23)$$

hence A can be immediately determined:

$$A^T = JAJ. \quad (24)$$

Thus, A is an element of $sp(4, R)$ algebra; and after the solution of (24), it can be represented as

$$A = \begin{pmatrix} -s^T & r \\ l & s \end{pmatrix}, \quad (25)$$

where $l^T = l$, $r^T = r$ and s are the two-dimensional matrices which in sum define ten independent parameters.

Then, in order that $M^T = M$ followed from $M_0^T = M_0$, the evolutionary operator should satisfy the (nongroup) condition

$$S^T = M_0^{-1} S M_0, \quad (26)$$

which imposes on A the following restriction:

$$A^T = M_0^{-1} A M_0. \quad (27)$$

Here we can finish the general analysis and turn our attention to the solutions determined by isotropic geodesic lines in the target space. From (19)–(21) follows that condition $dS^2 = 0$ is equivalent to

$$Tr A^2 = 0. \quad (28)$$

Then from (17) and (22) immediately follows that h_{ik} and $\lambda(x^i)$ can be taken in the form

$$h_{ik} = \delta_{ik}, \quad \lambda = \sum \frac{\lambda_n}{|\vec{r} - \vec{r}_n|} \quad (29)$$

where \vec{r} is as usual connected to x^i and \vec{r}_n is considered as the position of the n -center characterized by λ_n . We will assume that $\sum \lambda_n \neq 0$ (the dropped special case can be investigated in the same way), then in view of (21), without loss of generality, it is possible to impose on λ_n the normalization condition

$$\sum_n \lambda_n = 1. \quad (30)$$

It is evident that $\lambda \rightarrow 0$ when $r \rightarrow \infty$, thus $M_0 = M_\infty$. Let us naturally determine the asymptotic values of physical fields assuming that

$$f_\infty = 1, \quad \chi_\infty = u_\infty = v_\infty = \phi_\infty = \kappa_\infty = 0; \quad (31)$$

then from (11)–(13) we obtain

$$M_0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \equiv \Sigma_3, \quad (32)$$

and σ_3 is one of the Pauli matrices. By substituting the found value M_0 in (27), A can be calculated as

$$A = \begin{pmatrix} -\tilde{s} & r \\ \tilde{r} & s \end{pmatrix}, \quad (33)$$

where $s^T = \tilde{s}$ and for any two-dimensional matrix m we define $\tilde{m} = \sigma_3 m \sigma_3$.

From (21), (29), (30) it follows that at $r \rightarrow \infty$

$$S \longrightarrow I + \frac{A}{r} \quad (34)$$

and because of (20)

$$M \longrightarrow \Sigma_3 + \frac{A \Sigma_3}{r}. \quad (35)$$

After that, applying (11)–(13) and (25) it is easy to show that the main parts of the asymptotic decomposition of the functions $f - 1, \chi, u, v, \phi, \kappa$ are proportional to $\frac{1}{r}$. In this case, six components of matrices s and r act as coefficients which in this way determine six physical charges of the system. By entering the mass M , the parameter NUT N and also the electric Q_e , magnetic Q_m , dilaton D and axion A charges according to formulae

$$\begin{aligned} f &\rightarrow 1 - \frac{2M}{r}, & \chi &\rightarrow \frac{2N}{r}, \\ v &\rightarrow \frac{\sqrt{2}Q_e}{r}, & u &\rightarrow \frac{\sqrt{2}Q_m}{r}, \\ \phi &\rightarrow \frac{D}{r}, & \kappa &\rightarrow \frac{2A}{r}, \end{aligned} \quad (36)$$

s and r are found as follows:

$$s = \begin{pmatrix} -2M & \sqrt{2}Q_e \\ -\sqrt{2}Q_e & -2D \end{pmatrix}, \quad (37)$$

$$r = \begin{pmatrix} 2N & -\sqrt{2}Q_m \\ -\sqrt{2}Q_m & -2A \end{pmatrix}. \quad (38)$$

Let us determine now the evident form of the evolutionary operator S , which was written before with the aid of the formal exponential series. The calculation of A^2 in view of (33) gives:

$$A^2 = \alpha^\mu T_\mu, \quad (39)$$

where parameters α^μ are of the second order with respect to charges, T_0 is the unit matrix and three traceless matrices T_i are

$$T_1 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad T_3 = \Sigma_3. \quad (40)$$

It is convenient to unite six real charges into three complex ones:

$$\begin{aligned} \mathcal{M} &= M + iN, \\ \mathcal{D} &= D + iA, \\ \mathcal{Q} &= Q_e + iQ_m, \end{aligned} \quad (41)$$

in terms of which

$$\begin{aligned} \alpha^0 &= 2(\bar{\mathcal{M}}\mathcal{M} + \bar{\mathcal{D}}\mathcal{D} - \bar{\mathcal{Q}}\mathcal{Q}), \\ \alpha^1 + i\alpha^2 &= -2\sqrt{2}(\mathcal{M}\bar{\mathcal{Q}} + \bar{\mathcal{D}}\mathcal{Q}), \\ \alpha^3 &= 2(\bar{\mathcal{M}}\mathcal{M} - \bar{\mathcal{D}}\mathcal{D}). \end{aligned} \quad (42)$$

In doing so the isotropic condition (28) can be rewritten as

$$\bar{\mathcal{M}}\mathcal{M} + \bar{\mathcal{D}}\mathcal{D} = \bar{\mathcal{Q}}\mathcal{Q} \quad (43)$$

and generalizes the known relations in the Einstein-Maxwell theory [20]–[22].

It is easy to verify that the commutators of the matrices T_i are not their linear combinations, i.e., these matrices do not form the basis of a three-dimensional Lie algebra. But the calculation of the corresponding anticommutators leads to the following result:

$$\{T_i, T_j\} = T_i T_j + T_j T_i = -\eta_{ij}, \quad (44)$$

where $\eta_{ij} = \text{diag}(1, 1, -1)$, thus in view of (39) and (40) it appears that

$$A^4 = -\eta_{ij}\alpha^i\alpha^j. \quad (45)$$

The application of the relation (43) also allows to determine the fact that the quadratic form $\eta_{ij}\alpha^i\alpha^j$ is not negative and enter a new parameter α according to the definition

$$\alpha^4 = \eta_{ij}\alpha^i\alpha^j \quad (46)$$

The relation (46) allows to sum the series which correspond to the items with $n = 4k$ from the exponent decomposition (21). Now it is not difficult to find the evident form for the remaining three series with $n = 4k+1$, $n = 4k+2$ and $n = 4k+3$. The mentioned four series just compose the evolutionary operator and its expression in terms of the charge matrix A , defined by (33), (37), (38) and (43), and by the function λ (29) is

$$S = \sum_0^3 S_\mu A^\mu, \quad (47)$$

where A^μ is the matrix A to the μ power and

$$\begin{aligned} S_0 &= \cosh(\alpha\lambda/\sqrt{2}) \cos(\alpha\lambda/\sqrt{2}), \\ S_1 &= \frac{1}{\sqrt{2}\alpha} (\sinh(\alpha\lambda/\sqrt{2}) \cos(\alpha\lambda/\sqrt{2}) \\ &\quad + \cosh(\alpha\lambda/\sqrt{2}) \sin(\alpha\lambda/\sqrt{2})), \\ S_2 &= \frac{1}{\alpha^2} \sinh(\alpha\lambda/\sqrt{2}) \sin(\alpha\lambda/\sqrt{2}), \\ S_3 &= \frac{1}{\sqrt{2}\alpha^3} (\cosh(\alpha\lambda/\sqrt{2}) \sin(\alpha\lambda/\sqrt{2}) \\ &\quad - \sinh(\alpha\lambda/\sqrt{2}) \cos(\alpha\lambda/\sqrt{2})). \end{aligned} \quad (48)$$

The constructed solution (29) and (47)–(48) defines the system of interacting point centers which satisfies the restriction (43).

Let us turn our attention to the group nature of the matrix S . From (33) it is evident that the determining S matrix A differs from the belonging to the $sp(4, R)$ algebra general matrix by

$$\Gamma = \begin{pmatrix} \tilde{\tau} & \rho \\ -\tilde{\rho} & \tau \end{pmatrix}, \quad (49)$$

where $\tilde{\tau} = -\tau^T$, and ρ is symmetric. Entered here Γ has four independent parameters, the corresponding linear independent matrices (generators) can be written as

$$\begin{aligned} \Gamma_0 &= \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \Gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \\ \Gamma_2 &= \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \Gamma_3 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \end{aligned} \quad (50)$$

It is easy to prove that

$$[\Gamma_0, \Gamma_i] = 0 \quad (51)$$

and pair products for Γ_i are

$$\Gamma_i \Gamma_j = I \eta_{ij} + \epsilon_{ijk} \eta^{kl} \Gamma_l \quad (52)$$

where $\eta^{kl} = \eta_{kl}$.

Because of the resulting from (52) commutation relations, the isomorphism between algebra of Γ_i and that of two-dimensional Pauli matrices σ_i

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (53)$$

belonging to $sl(2, R)$, can be determined as

$$\Gamma_1 \sim \sigma_1, \quad \Gamma_2 \sim \sigma_3, \quad \Gamma_3 \sim \sigma_2. \quad (54)$$

If we also identify $\Gamma_0 \sim I$, it is easy to notice that the algebra of matrices T_μ appears to be isomorphic to $sl(2, R) \oplus R \sim gl(2, R)$. So the part of the group omitted while constructing the evolutionary operator is locally isomorphic to $GL(2, R)$ and hence $S \in Sp(4, R)/GL(2, R)$.

Later on it can be seen that the matrix

$$G = e^\Gamma = e^{\gamma^\mu \Gamma_\mu} \quad (55)$$

in view of (49) satisfies the relation

$$G^T \Sigma_3 G = \Sigma_3. \quad (56)$$

This allows to interpret G as a general matrix of belonging to $Sp(4, R)$ transformation which preserves the asymptotical vacuum values for the system of physical fields.

It is necessary to remark that the formal expression (55) as the corresponding one for S can be obtained in the evident form. Namely, let us denote the matrix constructed on Γ_0 by $G_{(0)}$ and that of constructed on Γ_i by $G_{(3)}$. Then in view of (51)

$$G = G_{(0)} G_{(3)} = G_{(3)} G_{(0)}. \quad (57)$$

Employing the relation

$$\Gamma_0^2 = -I \quad (58)$$

we have that

$$G_{(0)} = I \cos \gamma^0 + \Gamma_0 \sin \gamma^0. \quad (59)$$

Then, noticing that from (52) follows the anticommutation relations $\{\Gamma_i \Gamma_j\} = 2I \eta_{ij}$ the expression for $G_{(3)}$ matrix can be found:

$$\begin{aligned} 2G_{(3)} = & I[(1 + \sigma) \cosh \gamma + (1 - \sigma) \cos \gamma] \\ & + \frac{\gamma^i \Gamma_i}{\gamma} [(1 + \sigma) \sinh \gamma + (1 - \sigma) \sin \gamma] \end{aligned} \quad (60)$$

where the parameter γ is determined as

$$(\gamma)^2 = \sigma \eta_{ij} \gamma^i \gamma^j \quad (61)$$

and $\sigma = \text{sign}(\eta_{ij} \gamma^i \gamma^j)$.

IV. BLACK HOLES AND NAKED SINGULARITIES

The general geodesic isotropic solution of the string gravity obtained in the previous part has intricate dependence from the function λ , which satisfies Laplace equation and hence, from the space coordinates. It is easy to verify that in the case when $\alpha = 0$, i.e., if

$$\alpha_i \alpha_j \eta^{ij} = 0 \quad (62)$$

the evolutionary operator S becomes the polynomial of third power on λ which considerably facilitates the solution analysis. The greatest simplification is obtained at the simultaneous impose of the set of three additional relations $\alpha^i = 0$ to the physical charges, which according to (42) are equivalent to

$$\begin{aligned} \mathcal{M}\bar{\mathcal{Q}} + \mathcal{Q}\bar{\mathcal{D}} &= 0, \\ \mathcal{M}\bar{\mathcal{M}} - \mathcal{D}\bar{\mathcal{D}} &= 0. \end{aligned} \quad (63)$$

In doing so, in view of (21) and (39) the evolutionary operator and the null-curvature matrix M occur to be the linear functions of λ and satisfy, according to (17), the Laplace equation. The result can be investigated and, as it is further demonstrated, it contains interesting physical solutions. At first and foremost the independent ‘coordinates’ can be entered in the charge space. The number of such ‘coordinates’ appears to be equal to three, as it immediately follows from (43) and (63). By applying the complex form of transcription (41) we have:

$$\begin{aligned} \mathcal{M} &= \rho e^{2i\delta_1}, & \mathcal{D} &= \rho e^{2i\delta_2}, \\ \mathcal{Q} &= -i\sqrt{2}\sigma\rho e^{i(\delta_1+\delta_2)}, \end{aligned} \quad (64)$$

or, going to the real charges

$$\begin{aligned} M &= \rho \cos 2\delta_1, \\ N &= \rho \sin 2\delta_1, \\ D &= \rho \cos 2\delta_2, \end{aligned}$$

$$A = \rho \sin 2\delta_2, \quad (65)$$

$$Q_e = \sqrt{2}\sigma\rho \sin(\delta_1 - \delta_2),$$

$$Q_m = -\sqrt{2}\sigma\rho \cos(\delta_1 - \delta_2),$$

and $\sigma = \pm 1$. Turning back to relation (20) it is possible to calculate the matrix M . After that, employing the Gauss decomposition formula (11) and formulae (12) and (13), which determine the explicit dependance of matrix elements from six independent σ -model functions, the expressions can be found as follows:

$$\begin{aligned} f &= (1 + 2M\lambda)^{-1}, \\ \chi &= -2N\lambda(1 + 2M\lambda)^{-1}, \\ v &= \sqrt{2}Q_e\lambda(1 + 2M\lambda)^{-1}, \\ u &= \sqrt{2}Q_m\lambda(1 + 2M\lambda)^{-1}. \end{aligned} \quad (66)$$

In this case the expressions for axion and dilaton appear to be rather cumbersome, but by entering the complex variable

$$z = \kappa + ie^{-2\phi} \quad (67)$$

which, according to [10], is one of the Ernst potentials for the stationary system (6)–(7) the following compact result, which generalizes expressions found in [13] and [14], can be obtained

$$z = i \frac{1 + \lambda(\mathcal{M} - \mathcal{D})}{1 + \lambda(\mathcal{M} + \mathcal{D})}. \quad (68)$$

In this case, when the solution is determined only by one center, i.e., when $\lambda = \frac{1}{r}$, it is convenient to turn to a new radial coordinate $R = r + 2M$. It is easy to show that the expressions for electric and magnetic potentials transform to the most simple Coulomb form

$$\begin{aligned} v &= \frac{\sqrt{2}Q_e}{R}, \quad u = \frac{\sqrt{2}Q_m}{R}, \\ z &= i \frac{R - \bar{\mathcal{M}} - \mathcal{D}}{R - \bar{\mathcal{M}} + \mathcal{D}}. \end{aligned} \quad (69)$$

The employment of (5) and (66) allows to determine the obvious form of four-dimensional space-time metric:

$$ds^2 = \left(1 - \frac{2M}{R}\right)(dt - 2N\cos\theta d\phi)^2 - \left(1 - \frac{2M}{R}\right)^{-1}dR^2 - R(R - 2M)d\Omega^2. \quad (70)$$

In the constructed solution the mass, which causes the horizon appearance at $R = R_H = 2M$, and the parameter NUT which makes the spatial interval asymptotically different from Minkowski metric, appear to be independent parameters. This means that exist special solutions, which are asymptotically flat and have the horizon (black holes) and also solutions, possessing everywhere the regular but asymptotically unflat four-dimensional metric with the Coulomb-like expressions for the material fields (the naked singularities). It is important to note that the presence of the naked singularities in the string gravity appears to be possible due to the existence of the scalar sector in the theory, i.e., dilaton and axion fields.

From here on while investigating the special cases, both the results for the multicenter system and formulae (69) and (70) describing isolated singular object, will be taken into account. So, let us discuss the case $N = 0$, that according to (65) and condition $M > 0$ is equivalent to the relation $\delta_1 = \pi n$. The corresponding formulae for the charges lead to the following expressions:

$$D \sim Q_m^2 - Q_e^2, \quad A \sim Q_m Q_e \quad (71)$$

It is seen that the dilaton black holes (with $A = 0$) have one of the electromagnetic charges equal to zero while the axion black holes (for them $D = 0$) have equal in absolute magnitude electric and magnetic charges. Hence from the constructed before family of solutions (66), (68) and (69)–(70) naturally four black holes subfamilies stand out: dilaton magnetic, for which $\delta_2 = \pi k$,

$$D = M, \quad Q_m = \sqrt{2}\sigma M, \quad (72)$$

and all the other charges are equal to zero; dilaton electric, for which $\delta_2 = \pi(k + 1/2)$

$$D = -M, \quad Q_e = \sqrt{2}\sigma M; \quad (73)$$

axion with $Q_e = -Q_m$ when $\delta_2 = \pi(k + 1/4)$

$$A = M, \quad Q_e = -Q_m = \sigma M; \quad (74)$$

and, at last, axion with $Q_e = Q_m$ appearing at $\delta_2 = \pi(k + 3/4)$ and possessing

$$A = -M, \quad Q_e = Q_m = \sigma M. \quad (75)$$

One can notice that the discrete transformation $Q_m \rightarrow Q_e, \quad Q_e \rightarrow -Q_m$ transforms magnetic dilaton solution (72) into electric one (73) with simultaneous change $D \rightarrow -D$. In doing so the appropriate axion solutions transform one into another, if besides of the above mentioned electromagnetic transformation $A \rightarrow -A$ taking place.

Now we can study the massless solution families having, in accordance with (65) the NUT parameter value not equal to zero. From (70) it can be seen that the space metric is regular everywhere and only matter fields have physical peculiarities. So let us examine the case $M = 0$. As the parameter N can be of different sign, from (65) follows that $\delta_1 = \pi(2n + 1)/4$. Omitting the technical details we will turn our attention to the discussion of the main results. It turns out that the family of naked singularities under investigation, just as described above the black holes family, has the four most simple solution classes. Namely, the case when $\delta_1 = \pi(l + 1/4), \delta_2 = \pi(k + 1/4)$ and $\delta_1 = \pi(l + 3/4), \delta_2 = \pi(k + 3/4)$ corresponds to the axion magnetic solution (constructed earlier in [9]):

$$A = N, \quad Q_m = \sqrt{2}\sigma N; \quad (76)$$

when $\delta_1 = \pi(l + 1/4), \delta_2 = \pi(k + 3/4)$ and $\delta_1 = \pi(l + 3/4), \delta_2 = \pi(k + 1/4)$ we get dilaton electric singularity:

$$A = -N, \quad Q_e = \sqrt{2}\sigma N; \quad (77)$$

when $\delta_1 = \pi(l + 3/4), \delta_2 = \pi(k + 1/2)$ and $\delta_1 = \pi(l + 1/4), \delta_2 = \pi k$ it appears that

$$D = N, \quad Q_e = -Q_m = \sigma N, \quad (78)$$

and we get dilaton singularity with electromagnetic charges of different sign; and, finally, when $\delta_1 = \pi(l + 1/4)$, $\delta_2 = \pi(k + 1/2)$ and $\delta_1 = \pi(l + 3/4)$, $\delta_2 = \pi k$ the fields configuration of the dilaton singularity is determined by equal values of electric and magnetic charges, which are connected with NUT parameter as follows:

$$D = -N, \quad Q_e = Q_m = \sigma N. \quad (79)$$

Similarly to the situation with black holes, the determined above discrete transformations acting in the charge space connect axion singularities (76) and (77) and also (with the corresponding change of axion charge to dilaton one) transfer dilaton solutions (78)–(79) one into another. It is important to stress the resulting from formulae (72)–(75) and (76)–(79) formal analogy between dilaton black holes and naked axion singularities from one hand and axion black holes and dilaton naked singularities from the other. Turning back to the solution (65), (69), (70) describing singular object with mass and parameter NUT it can be pointed out that the solutions describing asymptotic flat black holes can be transformed into solutions for the horizonless asymptotic nonflat naked singularities with the aid of continuous transformation of the parameter δ_1 .

V. CONCLUSION

Using the Kramer–Neugebauer method for the null–curvature matrix $\text{Sp}(4, \mathbb{R})/\text{U}(2)$ coset formulation of the stationary D=4 EMDA theory we have constructed a new class of solutions which describe a system of interacting point centers. These centers describe a set of Majumdar–Papapetrou–like black holes in a special case and massless naked singularities in another one. A general class is connected with a complete family of isotropic geodesic lines which are crossing in a Minkowski vacuum point of the target space. As it has been shown, the evolutionary operator transforming vacuum solution to nontrivial one belongs to

$\mathrm{Sp}(4,\mathbb{R})/\mathrm{GL}(2,\mathbb{R})$ coset. The omitted four generators of $\mathrm{GL}(2,\mathbb{R})$ subgroup defines the general automorphism for the $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$ target space which preserves asymptotic flatness.

Used formalism admits the natural generalization when solutions are defined by extremal area surfaces in the potential space. It gives the possibility to construct Israel–Wilson–like sources for the theory under consideration.

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